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[www.elsevier.com/locate/jnt](http://www.elsevier.com/locate/jnt)Solving the Diophantine equation  $y^2 = x(x^2 - n^2)$ Konstantinos Draziotis<sup>a</sup>, Dimitrios Poulakis<sup>b,\*</sup><sup>a</sup> Kromnis 33, 54454 Thessaloniki, Greece<sup>b</sup> Aristotle University of Thessaloniki, Department of Mathematics, 54124 Thessaloniki, Greece

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## ABSTRACT

In this paper we give some necessary conditions satisfied by the integer solutions of the Diophantine equation  $y^2 = x(x^2 - n^2)$ . Next, using these conditions, we develop an algorithm for solving the equation.

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## 1. Introduction

Let  $n$  be a positive integer and  $E_n$  be the elliptic curve defined by the equation

$$y^2 = x(x^2 - n^2). \quad (1)$$

Eq. (1) has always the integer solutions  $(0, 0)$ ,  $(\pm n, 0)$  which we shall call trivial. These couples and the point at infinity are all the torsion points of  $E_n$  over  $\mathbb{Q}$ . In [8, Lemma 1.1], some sufficient conditions on  $n$  are listed for the rank of  $E_n(\mathbb{Q})$  to be zero. Furthermore, if  $n$  is a perfect square, then, it is easily seen, that the rank of  $E_n(\mathbb{Q})$  is zero. The integer points of  $E_n$  can be determined by the elliptic logarithm method [11,17], if one knows a full set of generators for the group  $E_n(\mathbb{Q})$ , modulo torsion. Algorithms for finding such generators exist but are not guaranteed to give always an answer [5,10]. Finally, note that the elliptic curves  $E_n$  are closely related with the congruent number problem (see [3] and [13]).

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In [6], the integer solutions of (1), in the case where  $n = p^a$  with  $p$  prime, are determined. When  $n = 2^a p^b$  where  $p$  is an odd prime and  $a > 0$ ,  $b > 0$ , an algorithm is given in [7] for the calculation of the integer solutions of (1) which does not use the structure of the group  $E_n(\mathbb{Q})$ . The aim of this paper is to generalize the approach of [7] and develop an algorithm for the determination of the integer solutions of (1), for any  $n$ . We use the “multiplication by 2” on  $E_n$  and we reduce the problem of the determination of integer solutions of (1) to the solution of a finite number of unit equations over some bicyclic biquadratic fields.

Let  $a$  be an integer and  $p$  be a prime. We denote by  $\text{ord}_p(a)$  the greatest integer  $r \geq 0$  such that  $p^r$  divides  $a$ . Let  $\Sigma_n$  be the set consisting of the following positive integers:

1. The integers of the form  $(a - b)^2/4$ , where  $a, b$  are positive integers with  $ab = n$ .
2. The integers of the form  $(a + b)/2$ , where  $a, b$  are positive integers with  $ab = n^2$ .
3. The integers of the form  $d(a \pm b)^2/4$ , where  $d$  is a positive divisor of  $n$  such that  $\text{ord}_p(d) = \text{ord}_p(n)$  for every odd prime divisor  $p$  of  $d$  with odd exponent and  $a, b$  are positive integers with  $ab = n/d$ .

Let  $D$  be the set of positive divisors of  $n$ ,  $d = d_1 \delta^2$ , where  $d_1, \delta \in \mathbb{Z}$  and  $d_1$  is square-free, such that for every prime divisor  $p$  of  $d_1$  we have  $\text{ord}_p(d) = \text{ord}_p(n)$ . For every element  $d = d_1 \delta^2$  of  $D$  we consider the negative integers  $x$  with  $|x| < n$ ,  $x = -daA^2$ , where  $a, A \in \mathbb{Z}$ ,  $a \perp d_1$ , for every odd prime divisor  $q$  of  $n/d$  we have  $-(d_1/a)/q = 1$  and  $a((n/d)^2 - (x/d)^2)/d_1$  is a square ( $(\cdot/q)$  denotes the usual Legendre symbol). We denote the set of all these integers by  $\Lambda_n$ .

Let  $K$  be a number field of degree  $k$ . For  $a \in K$  we denote by  $\Delta(a)$  the discriminant of the elements  $1, a, \dots, a^{k-1}$ . Furthermore, we denote by  $N_K$ ,  $\text{Tr}_K$  and  $O_K$  the norm, the trace and the ring of integers of  $K$ , respectively.

**Theorem 1.** Let  $(x, y) \in \mathbb{Z}^2$  be a nontrivial integer solution of (1) with  $x \notin \Sigma_n \cup \Lambda_n$ . Then  $x > n$  and there is a divisor  $d$  of  $n$  with  $d = wuvA^2$ , where  $A \in \mathbb{Z}$  and  $w, u, v$  are pairwise relatively prime, square-free, positive integers such that the field  $K = \mathbb{Q}(\sqrt{uv}, \sqrt{2^e wv})$ , where  $e \in \{0, 1\}$ , has degree 4 and contains a primitive element  $r \in O_K$  such that

$$x = \frac{d(r^2 + m^2)^2}{4r(r^2 - m^2)},$$

where  $m = n/d$ . We have  $N_K(r) = m^4$ ,  $N_K(r \pm m) = 4m^4$  and  $\text{Tr}_K(r) = 4x/d$ . If  $\sqrt{2} \in K$ , then  $(r \pm m)/\sqrt{2} \in O_K$  and  $N_K((r \pm m)/\sqrt{2}) = m^4$ . The numbers  $w, u$ , and  $v$  satisfy the following equalities:

1.  $\text{ord}_p(d) = \text{ord}_p(n)$ , for every odd prime divisor  $p$  of  $wuv$ .
2.  $(wu2^e/p) = 1$  and  $(wv2^e/p) = 1$ , for every odd prime divisor  $p$  of  $m$ .
3.  $(mu2^e/p) = 1$  and  $(-mv2^e/p) = 1$ , for every odd prime divisor  $p$  of  $w$ .
4.  $(-wm/p) = 1$  and  $(-mv2^{e+1}/p) = 1$ , for every odd prime divisor  $p$  of  $u$ .
5.  $(wm/p) = 1$  and  $(mu2^{e+1}/p) = 1$ , for every odd prime divisor  $p$  of  $v$ .

Furthermore, there is a primitive element  $\eta \in O_K$  satisfying  $\Delta(\eta) = 2^{14}n^6$ ,  $\text{Tr}_K(\eta) = 0$  and

$$(\eta^4 - 6x\eta^2 - 3x^2 + 4n^2)^2 - 64\eta^2(x^3 - n^2x) = 0.$$

Moreover, if  $d = n$ , then  $x = nwA^2$ , where  $(A, B)$  is an integer solution to the equation  $w^2X^4 - uvY^2 = 1$ .

Let  $R$  and  $S$  be maximal sets of pairwise nonassociate  $z \in O_K$  with  $N_K(z) = m^4$  and  $N_K(z) = 4m^4$ , respectively. By [2, Theorem 5, p. 90],  $R$  and  $S$  are finite. Since  $N_K(r) = m^4$  and  $N_K(r \pm m) = 4m^4$ ,

there are  $a \in R$ ,  $b \in S$  and units  $u, v \in K$  such that  $r = au$  and  $r - m = bv$ . It follows that  $u, v$  is a solution of the unit equation

$$aU - bV = m.$$

If  $\sqrt{2} \in K$ , then there are  $a, b \in R$  and units  $u, v \in K$  such that  $r = au$  and  $(r - m)/\sqrt{2} = bv$ . Thus,  $u, v$  is a solution of the unit equation

$$aU - b\sqrt{2}V = m.$$

Note that the sets  $R$  and  $S$  can be easily determined using the computational algebraic systems KANT/KASH [20] and MAGMA [21]. An efficient method for the solution of the unit equation is the Wildanger's algorithm [18]. It is also implemented in the systems KANT/KASH and MAGMA and so provides easily the solutions of the above equations.

On the other hand, [12] implies that there exists an effectively determined finite set  $T \subset O_K$  such that for every  $v \in O_K$  with  $\Delta(v) = 2^{16}n^6$  there are  $w \in T$  and  $z \in \mathbb{Z}$  with  $v = w + z$ . Thus, for the algebraic integer  $\eta$  of Theorem 1 there are  $b \in T$  and  $k \in \mathbb{Z}$  such that  $\eta = b + k$ . Since  $\text{Tr}_K(\eta) = 0$ , we have  $\text{Tr}_K(b) = -4k$ . Hence,  $\eta$  is among the primitive elements  $b + k$  with  $b \in T$ ,  $k \in \mathbb{Z}$  and  $\text{Tr}_K(b) = -4k$ . Note that the solvability of the discriminant form equation is equivalent to the solvability of the index form equation  $I_K(\eta) = 2^7n^3/\sqrt{D_K}$ . A method for the solution of the index form equation is implemented in the systems KANT/KASH and MAGMA.

Finally, the equation  $b^2x^4 - dy^2 = 1$  can be solved using the results of [4] and [1].

Taking in account the efficiency of the implemented algorithms, we combine the above approaches and we develop an algorithm for the solution of Eq. (1). Firstly, for every divisor  $d$  of  $n$ , using the relations (1)–(5) of Theorem 1, we obtain the possible cases for the field  $K$  and checking the solvability of the index form equation  $I_K(\eta) = 2^7n^3/\sqrt{D_K}$  we reduce further their number. Next, for every such field  $K$ , we determine maximal sets  $R$  and  $S$  of pairwise nonassociate  $z \in O_K$  with  $N_K(z) = m^4$  and  $N_K(z) = 4m^4$ , respectively, and solving a family of unit equations  $aU - bV = m$ , where  $(a, b) \in R \times S$ , we find the solutions of (1) (if  $\sqrt{2} \in K$ , then we solve unit equations  $aU - b\sqrt{2}V = m$ , where  $a, b \in R$ ). In case where  $d = n$  we can alternatively solve some equations of the form  $b^2x^4 - dy^2 = 1$ .

The paper is organized as follows. In Section 2 we give some auxiliary results useful for the proof of Theorem 1. Section 3 is devoted in the proof of Theorem 1. For the convenience of the reader, some results on the index form equations and the equation  $b^2x^4 - dy^2 = 1$  are presented in Sections 4 and 5, respectively. In Section 6 we specialize our results in case of integers  $n$  with only two odd prime divisors. Finally, in Section 7, we give an algorithm for the case where  $n$  has at least two prime factors.

## 2. Auxiliary lemmata

In this section we give some lemmata which are useful for the proof of Theorem 1.

**Lemma 1.** *Let  $(x, y) \in \mathbb{Z}^2$  be an integer solution of (1) with  $y \neq 0$  and  $d = \gcd(x, n)$ . If  $d = d_1\delta^2$ , where  $d_1, \delta$  are positive integers and  $d_1$  is square-free, then for every prime divisor  $p$  of  $d_1$  we have  $\text{ord}_p(d) = \text{ord}_p(n)$ .*

**Proof.** Since  $d = \gcd(x, n)$ , we have  $x = dz$  and  $n = dm$ , where  $z$  and  $m$  are positive integers with  $\gcd(z, m) = 1$ . Let  $p$  be a prime divisor of  $d_1$  with  $\text{ord}_p(d) < \text{ord}_p(n)$ . Then  $p \mid m$  and so  $p \nmid z$ . On the other hand, the equality

$$(y/d_1\delta^3)^2 = d_1z(z^2 - m^2)$$

implies that  $p \mid m^2 - z^2$  and so  $p \mid z$  which is a contradiction. Thus, for every prime divisor  $p$  of  $d_1$  we have  $\text{ord}_p(d) = \text{ord}_p(n)$ .  $\square$

**Lemma 2.** Let  $(x, y) \in \mathbb{Z}^2$  be an integer solution of (1) with  $x < 0$  and  $\gcd(x, n) = d$ . If  $d = d_1 \delta^2$ , where  $d_1, \delta$  are positive integers and  $d_1$  is square-free, then  $(n/d, d_1) = 1$ ,  $x = -daA^2$ , where  $a, A$  are positive integers with  $a \mid d_1$ , such that for every odd prime divisor  $q$  of  $n/d$  we have  $-(d_1/a)/q = 1$ . Furthermore,  $|x| \leq n$  and  $a((n/d)^2 - (x/d)^2)/d_1$  is a square.

**Proof.** We set  $x = -z$ , where  $z$  is a positive integer. Let  $\gcd(z, n) = d$  and  $d = d_1 \delta^2$ , where  $d_1, \delta$  are positive integers and  $d_1$  is square-free. We have  $z = dz_1$  and  $n = dm$ , where  $m$  and  $z_1$  are positive integers with  $\gcd(z_1, m) = 1$ . Since  $\gcd(z_1, m^2 - z_1^2) = 1$ , the equality

$$(y/d\delta)^2 = d_1 z_1 (m^2 - z_1^2)$$

implies that there are positive integers  $a, b, A, B$  with  $d_1 = ab$  such that

$$z_1 = aA^2, \quad m^2 - z_1^2 = bB^2.$$

Thus  $m^2 = z_1^2 + bB^2$  and  $\gcd(z_1, B) = 1$ . By Lemma 1, for every prime divisor  $p$  of  $d_1$ , we have  $\text{ord}_p(d) = \text{ord}_p(n)$  and so  $\gcd(d_1, m) = 1$ , whence  $\gcd(b, m) = 1$ . It follows that for every prime odd divisor  $q$  of  $m$  we have  $(-b/q) = 1$ .  $\square$

**Lemma 3.** Let  $(x, y) \in \mathbb{Z}^2$  be an integer solution of (1) such that  $x > n$ . Put  $\gcd(x, n) = d$ ,  $m = n/d$  and  $d = d_1 \delta^2$ , where  $\delta, d_1$  are positive integers and  $d_1$  is square-free. Then

$$x = dwA^2, \quad x + n = d2^e uB^2, \quad x - n = d2^e vC^2,$$

where  $w, u, v, A, B$  and  $C$  are positive integers with  $wuv = d_1$ . We have  $e = 1$  if  $m$  and  $x/d$  have the same parity and  $e = 0$  otherwise. Furthermore, the following equalities are satisfied:

1. For every odd prime divisor  $p$  of  $m$  we have

$$(wu2^e/p) = 1, \quad (wv2^e/p) = 1.$$

2. For every odd prime divisor  $p$  of  $w$  we have

$$(mu2^e/p) = 1, \quad (-mv2^e/p) = 1.$$

3. For every odd prime divisor  $p$  of  $u$  we have

$$(-wm/p) = 1, \quad (-mv2^{e+1}/p) = 1.$$

4. For every odd prime divisor  $p$  of  $v$  we have

$$(wm/p) = 1, \quad (mu2^{e+1}/p) = 1.$$

**Proof.** Putting  $x = dz$ , where  $z$  is an integer, we obtain

$$(y/d\delta)^2 = d_1 z(z^2 - m^2).$$

Since  $\gcd(z, m) = 1$ , we have  $\gcd(z, z \pm m) = 1$  and  $\gcd(z + m, z - m) = 2^e$ , where  $e = 1$ , if  $z$  and  $m$  have the same parity and  $e = 0$ , otherwise. It follows that

$$z = wA^2, \quad z + m = 2^e uB^2, \quad z - m = 2^e vC^2,$$

where  $w, u, v, A, B$  and  $C$  are positive integers with  $wuv = d_1$ . Finally, combining the above equalities, we obtain

$$wA^2 + m = 2^e uB^2, \quad wA^2 - m = 2^e vC^2, \quad 2m = 2^e uB^2 - 2^e vC^2,$$

whence we deduce the equalities (1)–(4).  $\square$

The following lemma is a version of Proposition 20 of [13]. We denote by  $\bar{\mathbb{Q}}$  an algebraic closure of  $\mathbb{Q}$ .

**Lemma 4.** *Let  $E$  be the elliptic curve  $Y^2 = (X - e_1)(X - e_2)(X - e_3)$  with  $e_1, e_2, e_3 \in \mathbb{Q}$ . If  $(x, y) \in \mathbb{Q}^2$  and  $(s, t) \in \bar{\mathbb{Q}}^2$  are two points on  $E$  such that  $(x, y) = [2](s, t)$ , then  $\mathbb{Q}(s, t) = \mathbb{Q}(\sqrt{x - e_i}, \sqrt{x - e_j})$ , for every pair  $i, j \in \{1, 2, 3\}$  with  $i \neq j$ .*

**Proof.** Consider the change of variables  $Z = X - x$ . We set  $w_i = e_i - x$  ( $i = 1, 2, 3$ ) and  $v = s - x$ . Then  $P = (0, y)$  and  $Q = (v, t)$  are two points on the elliptic curve  $E_1$  defined by  $Y^2 = (Z - w_1)(Z - w_2)(Z - w_3)$  such that  $P = [2]Q$ . The tangent line to  $E_1$  at  $Q$  passes through  $-P = (0, -y)$  and hence its equation is  $Y = mZ - y$ . If  $v = 0$ , then  $t = -y$  and we have  $m = 0$ . Suppose that  $v \neq 0$ . Then  $v$  is the double root of the cubic  $(mZ - y)^2 = (Z - w_1)(Z - w_2)(Z - w_3)$ . Since the third root of the cubic is zero, we have  $v = (w_1 + w_2 + w_3 + m^2)/2$ . Also, we have  $t = mv - y$ . Thus  $v, t \in \mathbb{Q}(m)$ . Conversely, since  $m = (t + y)/v$ , we have  $m \in \mathbb{Q}(v, t)$ . Therefore  $\mathbb{Q}(m) = \mathbb{Q}(v, t)$ .

Next, note that the points  $Q_i = Q \oplus (w_i, 0)$  ( $i = 1, 2, 3$ ) of  $E_1$  satisfy  $[2]Q_i = P$  and  $Q, Q_1, Q_2$  and  $Q_3$  are all the points of  $E_1$  having this property ( $\oplus$  denotes the addition on  $E$ ). Let  $Q_i = (v_i, t_i)$  ( $i = 1, 2, 3$ ). The tangent line to  $E_1$  at  $Q_i$  has equation  $Y = m_i Z - y$  ( $i = 1, 2, 3$ ). We have, as previously,  $\mathbb{Q}(m_i) = \mathbb{Q}(v_i, t_i)$ . The formulas of addition on  $E_1$  imply that  $v_i, t_i \in \mathbb{Q}(v, t)$  and so  $m_i \in \mathbb{Q}(m)$ . Hence  $\mathbb{Q}(m_i) \subseteq \mathbb{Q}(m)$ . Changing the role of  $m$  and  $m_i$  we have  $\mathbb{Q}(m) \subseteq \mathbb{Q}(m_i)$ . Therefore  $\mathbb{Q}(m_i) = \mathbb{Q}(m)$  ( $i = 1, 2, 3$ ).

Put  $m = m_0$ . For every  $i = 0, 1, 2, 3$  the equation

$$(m_i Z - y)^2 = Z^3 + aZ^2 + bZ + c,$$

where

$$a = -w_1 - w_2 - w_3, \quad b = w_1 w_2 + w_1 w_3 + w_2 w_3, \quad c = -w_1 w_2 w_3 = y^2$$

has a double root. Simplifying and factor out  $Z$ , we deduce that the equation

$$Z^2 + (a - m_i^2)Z + (b + 2m_i y) = 0$$

has a double root. This is equivalent to

$$(a - m_i^2)^2 - 4(b + 2m_i y) = 0 \quad (i = 0, 1, 2, 3).$$

Next, we introduce  $f_i$  satisfying  $f_i^2 = -w_i$  and  $f_1 f_2 f_3 = y$ . There are two possible choices for  $f_i$ , unless  $w_i = 0$ . If all the  $w_i$  are nonzero, then the condition  $f_1 f_2 f_3 = y$  implies that the signs of  $f_1$  and  $f_2$  can be arbitrary and the sign of  $f_3$  is chosen so that  $y$  and  $f_1 f_2 f_3$  are the same square root of  $-w_1 w_2 w_3$ . If, say,  $w_3 = 0$ , then either choice can be made for the sign of  $f_1, f_2$ , and  $f_3 = 0$ . Suppose, without loss of generality that  $w_1$  and  $w_2$  are nonzero. So, once we fix  $f_1, f_2, f_3$ , we have the following four choices:  $f_1, f_2, f_3$ ;  $f_1, -f_2, -f_3$ ;  $-f_1, f_2, -f_3$ ;  $-f_1, -f_2, f_3$ .

Setting  $s_1 = f_1 + f_2 + f_3, s_2 = f_1 f_2 + f_1 f_3 + f_2 f_3$  and  $s_3 = f_1 f_2 f_3$ , we get

$$a = s_1^2 - 2s_2, \quad b = s_2^2 - 2s_1 s_3, \quad y = s_3.$$

Thus, we have

$$\begin{aligned}(a - T^2)^2 - 4(b + 2Ty) &= (T^2 - s_1^2 + 2s_2)^2 - 4(s_2^2 - 2s_1s_3 + 2Ts_3) \\ &= (T^2 - s_1^2)^2 + 4s_2(T^2 - s_1^2) - 8s_3(T - s_1).\end{aligned}$$

It follows that  $s_1$  is a root of the above polynomial. Since we could have made three other choices for the signs of the  $f_i$ , the other roots must correspond to these choices. Hence, the roots of this polynomial are

$$M_1 = f_1 + f_2 + f_3, \quad M_2 = f_1 - f_2 - f_3, \quad M_3 = -f_1 + f_2 - f_3, \quad M_4 = -f_1 - f_2 + f_3$$

and of course we have  $\{m, m_1, m_2, m_3\} = \{M_1, M_2, M_3, M_4\}$ . We deduce that  $f_1 = (M_1 + M_2)/2$ ,  $f_2 = (M_1 + M_3)/2$  and  $f_3 = (M_1 + M_4)/2$ , whence  $\mathbb{Q}(f_1, f_2) \subseteq \mathbb{Q}(m)$ . Conversely, it is obvious that  $\mathbb{Q}(m) \subseteq \mathbb{Q}(f_1, f_2)$ . Hence  $\mathbb{Q}(m) = \mathbb{Q}(f_1, f_2)$ . The result follows.  $\square$

### 3. Proof of Theorem 1

By Lemma 1, we have  $x > 0$ . We consider  $(s, t) \in \tilde{\mathbb{Q}}^2$  such that  $[2](s, t) = (x, y)$ . Let  $K = \mathbb{Q}(s, t)$ . There are relatively prime integers  $z$  and  $m$  such that  $x = dz$  and  $n = dm$ . Set  $d = d_1\delta^2$ , where  $d_1, \delta \in \mathbb{Z}$  and  $d_1$  is square-free. By Lemmata 1 and 3, for every odd prime divisor  $p$  of  $d$  with odd exponent, we have  $\text{ord}_p(d) = \text{ord}_p(n)$  and

$$x = dwA^2, \quad x + n = d2^e uB^2, \quad x - n = d2^e vC^2,$$

where  $w, u, v, A, B$  and  $C$  are positive integers with  $wuv = d_1$ . Further,  $e = 1$  if  $m$  and  $z$  have the same parity and  $e = 0$  otherwise. Moreover, the following equalities are satisfied:

1. For every odd prime divisor  $p$  of  $m$  we have

$$(wu2^e/p) = 1, \quad (wv2^e/p) = 1.$$

2. For every odd prime divisor  $p$  of  $w$  we have

$$(mu2^e/p) = 1, \quad (-mv2^e/p) = 1.$$

3. For every odd prime divisor  $p$  of  $u$  we have

$$(-wm/p) = 1, \quad (-mv2^{e+1}/p) = 1.$$

4. For every odd prime divisor  $p$  of  $v$  we have

$$(wm/p) = 1, \quad (mu2^{e+1}/p) = 1.$$

Then Lemma 4 yields

$$K = \mathbb{Q}(\sqrt{uv}, \sqrt{2^e wv}) = \mathbb{Q}(\sqrt{uv}, \sqrt{2^e wu}) = \mathbb{Q}(\sqrt{2^e wu}, \sqrt{2^e wv}).$$

Suppose first that  $K = \mathbb{Q}$ . Then there are integers  $U, V$  and  $W$  such that  $x = U^2$ ,  $x + n = V^2$  and  $x - n = W^2$ . It follows that  $x = (\alpha - \beta)^2/4$ , where  $\alpha$  and  $\beta$  are positive integers with  $\alpha > \beta$  and  $\alpha\beta = n$ .

Next, we suppose that  $K$  is a quadratic field. We have the following cases:

1.  $uv = 1$ . Then  $u = v = 1$ . It follows that  $x = U^2$ , where  $U$  is positive integer, and Eq. (1) implies that there is a positive integer  $V$  such that  $V^2 = U^4 - n^2$ . Hence we have  $x = (\alpha + \beta)/2$ , where  $\alpha$  and  $\beta$  are positive integers with  $\alpha\beta = n^2$ .

2.  $2^e wv = 1$ . Then  $e = 0$ ,  $w = v = 1$ . We deduce that  $A^2 - m = C^2$ , whence we get  $A = (\alpha + \beta)/2$ , where  $\alpha$  and  $\beta$  are positive integers with  $\alpha\beta = m$ . Thus  $x = d(\alpha + \beta)^2/4$ .

3.  $uv \neq 1$  and  $2^e wv \neq 1$ . If  $d_1$  is odd, then  $e = 0$ ,  $u = w = 1$ . It follows that  $B^2 - A^2 = m$ , whence  $A = (\alpha - \beta)/2$ , where  $\alpha$  and  $\beta$  are positive integers with  $\alpha > \beta$  and  $\alpha\beta = m$  and so  $x = d(\alpha - \beta)^2/4$ . If  $d_1$  is even, then exactly one of  $w, u, v$  is even. Suppose that  $u$  is even. Then  $e = 1$ ,  $u = 2$  and  $w = 1$ . So, we have  $(2B)^2 - A^2 = m$  and we deduce that  $x$  is as in the previous case. If  $v$  is even, then we obtain  $A^2 - m = (2C)^2$  and we get  $x = d(\alpha + \beta)^2/4$ , where  $\alpha$  and  $\beta$  are positive integers with  $\alpha\beta = m$ . If  $w$  is even, then  $e = 1$ ,  $w = 2$  and  $v = 1$ . It follows that  $\gcd(z, m)$  is even which is a contradiction.

Finally, we suppose that  $[K : \mathbb{Q}] = 4$ . We have  $[2](s, t) = (\phi(s, t), \psi(s, t))$ , where

$$\phi(s, t) = -2s + \left(\frac{3s^2 - n^2}{2t}\right)^2, \quad \psi(s, t) = -t + \left(\frac{3s^2 - n^2}{2t}\right)(s - \phi(s, t)).$$

Putting  $\eta = (3s^2 - n^2)/2t$ , we get

$$x = -2s + \eta^2, \quad y = -\frac{3s^2 - n^2}{2\eta} + \eta(3s - \eta^2). \quad (2)$$

We eliminate  $s$  between these two equalities and we obtain that  $\eta$  satisfies the equation

$$h(U) = U^4 - 6xU^2 - 8yU - 3x^2 + 4n^2 = 0. \quad (3)$$

From the equality  $s = (\eta^2 - x)/2$  we obtain  $K \subseteq \mathbb{Q}(\eta)$  and since  $[\mathbb{Q}(\eta) : \mathbb{Q}] \leq 4$ , we have  $K = \mathbb{Q}(\eta)$ . Thus,  $h(U)$  is the minimal polynomial of  $\eta$ . The discriminant of  $h(T)$  is equal to  $2^{14}n^6$  and so  $\Delta(\eta) = 2^{14}n^6$ . Eliminating  $y$  between  $y^2 = x^3 - n^2x$  and (3), we obtain

$$(\eta^4 - 6x\eta^2 - 3x^2 + 4n^2)^2 = 64\eta^2(x^3 - n^2x).$$

Furthermore, we remark that  $\text{Tr}(\eta) = 0$ .

Next, substituting the values of  $x$  and  $y$  given by (2) in (3) and replacing  $\eta^2$  by  $2s + x$ , we obtain that  $s$  is a root of the equation

$$f(T) = T^4 - 4xT^3 + 2n^2T^2 + 4n^2xT + n^4 = 0. \quad (4)$$

We shall prove that  $K = \mathbb{Q}(s)$  and hence  $f(T)$  is the irreducible polynomial of  $s$ . Since  $s \neq 0$ , we have

$$0 = \frac{f(s)}{s^2} = \left(s - \frac{n^2}{s}\right)^2 - 4x\left(s - \frac{n^2}{s}\right) + 4n^2$$

and so, we get

$$s - \frac{n^2}{s} = 2(x \pm \sqrt{x^2 - n^2}),$$

whence

$$s^2 - 2(x \pm \sqrt{x^2 - n^2})s - n^2 = 0.$$

Therefore, we obtain

$$s = x \pm \sqrt{x^2 - n^2} \pm \sqrt{2x^2 \pm 2x\sqrt{x^2 - n^2}}.$$

We remark that  $[\mathbb{Q}(s) : \mathbb{Q}] < 4$  if and only if  $\sqrt{x^2 - n^2}$  is the square of an integer or  $2x^2 \pm 2x\sqrt{x^2 - n^2}$  is the square of an element of the ring of integers of  $\mathbb{Q}[\sqrt{x^2 - n^2}]$ . If  $x^2 - n^2 = a^2$ , where  $a \in \mathbb{Z}$ , then  $a^2 = uv d^2 2^{2e} B^2 C^2$ , whence  $uv = 1$  and so,  $[K : \mathbb{Q}] < 4$  which is a contradiction. Suppose that  $2x^2 \pm 2x\sqrt{x^2 - n^2} = (a + b\sqrt{x^2 - n^2})^2$ , where  $a, b \in \mathbb{Q}$ . It follows that  $2x^2 = a^2 + b^2(x^2 - n^2)$  and  $x = |a||b|$ . Eliminating  $x$ , we deduce  $a^2(b^2 - 1)^2 = n^2 b^2$ , whence  $|a|(b^2 - 1) = nb$ . Multiplying the two members of the last equality by  $|a|$ , we get  $a^2(b^2 - 1) = nx$ . Replacing  $ab$  by  $x$ , we obtain  $a^2 = x^2 - nx$ , whence we get  $a \in \mathbb{Z}$ . Thus, the equality  $a^2 = y^2/(x + n)$  implies that  $x + n$  is a perfect square. Since  $K = \mathbb{Q}(\sqrt{x}, \sqrt{x + n})$ , we deduce that  $[K : \mathbb{Q}] \leq 2$  which is a contradiction. Hence  $K = \mathbb{Q}(s)$  and  $f(T)$  is the irreducible polynomial of  $s$ .

Since  $r = s/d$  is a root of the equation

$$f_1(T) = T^4 - 4zT^3 + 2m^2T^2 + 4m^2zT + m^4 = 0,$$

we get  $N_K(r) = m^4$ . Further,  $r \pm m$  are roots of the irreducible polynomial

$$f_2(T) = f_1(T \mp m) = T^4 + 4(\mp m - z)T^3 + 4m(2m \pm 3z)T^2 + 8m^2(\mp m - z)T + 4m^4,$$

and so, we obtain  $N_K(r \pm m) = 4m^4$ . If  $\sqrt{2} \in K$ , then from the equation  $f_2(r \pm m) = 0$  we deduce that  $\sqrt{2}$  divides  $r \pm m$ . It follows that  $(r \pm m)/\sqrt{2}$  is an algebraic integer with  $N_K((r \pm m)/\sqrt{2}) = m^4$ . Further, from (4) we have

$$x = \frac{(s^2 + n^2)^2}{4s(s^2 - n^2)} = \frac{d(r^2 + m^2)^2}{4r(r^2 - m^2)}.$$

Finally, suppose that  $d = n$ . Then  $x = nwA^2$  and  $x^2 - n^2 = n^2uv(2^e BC)^2$ . It follows that  $w^2A^4 - uv(2^e BC)^2 = 1$ .

#### 4. Index form equations

Let  $a, b$  be two distinct square-free positive integers  $> 1$ . If  $l = \gcd(a, b)$ , then  $a = la_1$  and  $b = lb_1$ , where  $a_1, b_1$  pairwise prime positive integers. We consider the field  $K = \mathbb{Q}(\sqrt{a}, \sqrt{b})$ . We denote by  $D_K$  the discriminant of  $K$  and by  $I_K(x_2, x_3, x_4)$  the index form over  $K$ . Following [9] and [19], we list an integral basis, the discriminant and the index form for  $K$ . We have the following cases:

**Case 1.**  $a \equiv b \equiv a_1 \equiv b_1 \equiv 1 \pmod{4}$ . Integral basis:  $1, (1 + \sqrt{a})/2, (1 + \sqrt{b})/2, (1 + \sqrt{a} + \sqrt{b} + \sqrt{a_1b_1})/4$ . Discriminant:  $D_K = (la_1b_1)^2$ . Index form:

$$I_K(x_2, x_3, x_4) = (l(x_2 + x_4/2)^2 - b_1x_4^2/4)(l(x_3 + x_4/2)^2 - a_1x_4^2/4)(b_1(x_3 + x_4/2)^2 - a_1(x_2 + x_4/2)^2).$$

**Case 2.**  $a \equiv b \equiv 1 \pmod{4}, a_1 \equiv b_1 \equiv 3 \pmod{4}$ . Integral basis:  $1, (1 + \sqrt{a})/2, (1 + \sqrt{b})/2, (1 - \sqrt{a} + \sqrt{b} + \sqrt{a_1b_1})/4$ . Discriminant:  $D_K = (la_1b_1)^2$ . Index form:

$$I_K(x_2, x_3, x_4) = (l(x_2 - x_4/2)^2 - b_1x_4^2/4)(l(x_3 + x_4/2)^2 - a_1x_4^2/4)(b_1(x_3 + x_4/2)^2 - a_1(x_2 - x_4/2)^2).$$



**Case 3.**  $a \equiv 1 \pmod{4}$ ,  $b \equiv 2 \pmod{4}$ . Integral basis:  $1, (1 + \sqrt{a})/2, \sqrt{b}, (\sqrt{b} + \sqrt{a_1 b_1})/2$ . Discriminant:  $D_K = (4la_1 b_1)^2$ . Index form:

$$I_K(x_2, x_3, x_3) = (lx_2^2 - b_1 x_4^2)(l(x_3 + x_4/2)^2 - a_1 x_4^2/4)(4b_1(x_3 + x_4/2)^2 - a_1 x_2^2).$$

**Case 4.**  $a \equiv 2 \pmod{4}$ ,  $b \equiv 3 \pmod{4}$ . Integral basis:  $1, \sqrt{a}, \sqrt{b}, (\sqrt{a} + \sqrt{a_1 b_1})/2$ . Discriminant:  $D_K = (8la_1 b_1)^2$ . Index form:

$$I_K(x_2, x_3, x_3) = ((2x_2 + x_4)^2/2 - b_1 x_4^2/2)(2lx_3^2 - a_1 x_4^2/2)(2b_1 x_3^2 - a_1(2x_2 + x_4)^2/2).$$

**Case 5.**  $a \equiv b \equiv 3 \pmod{4}$ . Integral basis:  $1, \sqrt{a}, (\sqrt{a} + \sqrt{b})/2, (1 + \sqrt{a_1 b_1})/2$ . Discriminant:  $D_K = (4la_1 b_1)^2$ . Index form:

$$I_K(x_2, x_3, x_3) = (l(2x_2 + x_3)^2 - b_1 x_4^2)(lx_3^2 - a_1 x_4^2)(b_1 x_3^2/4 - a_1(x_2 + x_3/2)^2).$$

Consider now the index form equation

$$I_K(x_2, x_3, x_3) = \pm \mu,$$

where  $\mu$  is a positive integer.

**Proposition 1.** *If the above index form equation has a solution, then there are integers  $F_1, F_2, F_3$  with  $F_1 F_2 F_3 = \mu$  such that, according to Cases 1–5, we have:*

Case 1: The system  $lx^2 - b_1 y^2 = 4F_1$ ,  $lz^2 - a_1 y^2 = 4F_2$ ,  $b_1 z^2 - a_1 x^2 = 4F_3$  has an integer solution and  $-F_1 a_1 + F_2 b_1 = F_3 l$ .

Case 2: The system  $lx^2 - b_1 y^2 = 4F_1$ ,  $lz^2 - a_1 y^2 = 4F_2$ ,  $n_1 z^2 - a_1 x^2 = 4F_3$  has an integer solution and  $-F_1 a_1 + F_2 b_1 = F_3 l$ .

Case 3: The system  $lx^2 - b_1 y^2 = F_1$ ,  $lz^2 - a_1 y^2 = 4F_2$ ,  $b_1 z^2 - a_1 x^2 = F_3$  has an integer solution and  $-F_1 a_1 + 4F_2 b_1 = F_3 l$ .

Case 4: The system  $x^2 - b_1 y^2 = 2F_1$ ,  $2lz^2 - a_1/2 y^2 = F_2$ ,  $2b_1 z^2 - a_1/2 x^2 = F_3$  has an integer solution and  $-F_1 a_1 + F_2 b_1 = F_3 l$ .

Case 5: The system  $lx^2 - b_1 y^2 = F_1$ ,  $lz^2 - a_1 y^2 = F_2$ ,  $b_1 z^2 - a_1 x^2 = 4F_3$  has an integer solution and  $-F_1 a_1 + F_2 b_1 = 4F_3 l$ .

**Proof.** See [9, pp. 103–104].  $\square$

Note that we can check the solvability of Pell equations in Proposition 6 using the computational system MAPLE. Furthermore, in [14] elementary necessary and sufficient conditions for the solvability of the equation  $x^2 - Dy^2 = a$ , where  $D, a \in \mathbb{Z}$  and  $D > 0$ , are given.

## 5. The equation $b^2 x^4 - dy^2 = 1$

In this section, we give some results on the integer solutions of the above equation useful for the determination of the integer points of  $E_n$ . We call trivial solutions of this equation the solutions given by  $x^2 = 1$ .

**Theorem 2.** *Let  $a + b\sqrt{d}$  be the fundamental solution of the equation  $x^2 - dy^2 = 1$ . Then the only possible nontrivial integer solutions of the equation  $x^4 - dy^2 = 1$  are given by  $x^2 = a$  and  $x^2 = 2a^2 - 1$ ; both solutions occur in only one case,  $d = 1785$ .*

**Proof.** See [4].  $\square$

**Theorem 3.** *If  $d$  is a prime number or the double of a prime number, then the equation  $x^4 - dy^2 = 1$  has only the trivial solutions, unless  $d = 5, 29, 6$  in which cases there is also the solutions  $(x, y) = (\pm 3, \pm 4), (\pm 99, \pm 1820)$  and  $(\pm 7, \pm 20)$ , respectively.*

**Proof.** See [16, Th. 2.1].  $\square$

**Theorem 4.** *If  $p$  is a prime number, then the equation  $4x^4 - py^2 = 1$  has only the trivial solutions, unless  $p = 3, 7$  in which cases there is also the solutions  $(x, y) = (\pm 1, \pm 1)$  and  $(\pm 2, \pm 3)$ , respectively.*

**Proof.** See [16, Th. 2.2].  $\square$

Let now  $b$  and  $d$  be two square-free relatively prime integers  $> 1$ . We denote by  $\epsilon = T + U\sqrt{d}$  the fundamental solution to the Pell equation  $x^2 - dy^2 = 1$ , and, for  $k \geq 1$ , we put  $T_k + U_k\sqrt{d} = \epsilon^k$ . The divisibility index  $\alpha(b)$  of  $b$  in the sequence  $T_k$  is the smallest positive integer  $k$  for which  $b \mid T_k$ . If, for all  $k$ ,  $b$  do not divides  $T_k$ , then we set  $\alpha(b) = \infty$ .

**Theorem 5.** *There is at most one index  $k$  for which  $T_k = bx^2$  for some  $x \in \mathbb{Z}$ , and consequently, the equation  $b^2x^4 - dy^2 = 1$  has at most one solution in positive integers  $x, y$ . Moreover, if such solution exists, then  $k = \alpha(b)$ .*

**Proof.** See [1, Theorem 1.2].  $\square$

A useful corollary is the following:

**Corollary 1.** *Let  $b = 2^r 3^s 5^t 7^u 11^v$  for some integers  $r, s, t, u, v \in \{0, 1\}$ , not all zero. Then any solution of  $T_k = bx^2$  with  $x \in \mathbb{Z}$  implies  $k = 1$  unless*

1.  $b = 7$ , in which case  $k = 1$  or  $k = 2$  (but not both),
2.  $b = 11$  and  $d = 2$ , in which case  $T_3 = 11 \cdot 3^2$ ,
3.  $b = 55$  and  $d = 1139$ , in which case  $T_3 = 55 \cdot 423^2$ .

**Proof.** See [1, Corollary 1.3].  $\square$

The following proposition is stated without proof in [1, p. 3486]. Professor Gary Walsh kindly communicated to us its proof.

**Proposition 2.** *Suppose that  $p$  is a prime. If  $p = 2$ , then  $\alpha(p) = 1$  or  $\infty$  and if  $p > 2$ , then either  $\alpha(p) \mid (p - (d/p))/2$  or  $\alpha(p) = \infty$ .*

**Proof.** If  $p = 2$ , then it is easily seen that  $\alpha(p) = 1$  or  $\infty$ . Suppose that  $p > 2$ . If  $p$  divides  $d$  or  $U$ , then  $\alpha(p) = \infty$ . Suppose now that  $p$  does not divide  $d$  and  $U$ . We have  $U_n = u_n U$  and  $T_n = t_n/2$ , where  $u_n = (\epsilon^n - \epsilon^{-n})/(\epsilon - \epsilon^{-1})$  and  $t_n = \epsilon^n + \epsilon^{-n}$  are the Lucas sequences associated to polynomial  $X^2 - 2T + 1$  with discriminant equal to  $4(T^2 - 1) = 4dU^2$ . By [15, p. 44], we have  $u_{2k} = u_k t_k$ . Thus, if  $r$  is the highest power of 2 dividing  $p - (d/p)$ , then we have

$$u_{p-(d/p)} = u_{(p-(d/p))/2^r} t_{(p-(d/p))/2} \cdots t_{(p-(d/p))/2^r}.$$

By [15, p. 49],  $p \mid u_{p-(d/p)}$ . So, either  $p$  divides  $u_{(p-(d/p))/2^r}$  or  $p$  divides one of the  $t_{(p-(d/p))/2^i}$ . Suppose first that  $p \mid u_{(p-(d/p))/2^r}$ . If  $p$  divides some  $t_k$ , then  $p$  divides  $u_{2k} = u_k t_k$ . By [15, p. 51], we have  $\gcd(u_{2k}, u_{(p-(d/p))/2^r}) = u_{\gcd(2k, (p-(d/p))/2^r)}$  and so  $p$  divides  $u_{\gcd(2k, (p-(d/p))/2^r)}$ . Since  $(p - (d/p))/2^r$  is odd,  $\gcd(2k, (p - (d/p))/2^r)$  divides  $k$ . By [15, p. 47],  $u_{\gcd(2k, (p-(d/p))/2^r)}$  divides  $u_k$  and hence  $p \mid u_k$ . Further, [15, p. 51] implies  $\gcd(u_k, t_k) = 1$  or 2 and so  $p \nmid 2$  which is a contradiction.

Therefore,  $p$  does not divide  $t_k$  which implies  $\alpha(p) = \infty$ . Suppose next that  $p$  divides one of the  $t_{(p-(d/p))/2^i}$ . Since the set of  $k$  for which  $p \mid T_k$  is precisely the set  $\{t\alpha(p)/t \text{ is odd}\}$ , we deduce that  $\alpha(p)$  is a divisor of  $(p - (d/p))/2^i$  which is a divisor of  $(p - (d/p))/2$ . The result follows.  $\square$

**Proposition 3.** Let  $b$  be a square-free integer  $> 1$  and  $b = p_1 \cdots p_k$  the prime decomposition of  $b$ . If  $\alpha(b) < \infty$ , then  $\alpha(p_i) < \infty$  ( $i = 1, \dots, k$ ) and  $\alpha(b) = \text{lcm}(\alpha(p_1), \dots, \alpha(p_k))$ . Furthermore, if  $b$  is even, then  $\alpha(b)$  is odd.

**Proof.** The set of  $k$  for which  $p_i \mid T_k$  is precisely the set  $\{t\alpha(p_i)/t \text{ is odd}\}$  ( $i = 1, \dots, k$ ). Since  $\alpha(b) < \infty$ , then  $\alpha(p_i) < \infty$  ( $i = 1, \dots, k$ ) and there are odd integers  $s_1, \dots, s_k$  such that

$$\alpha(b) = \alpha(p_1)s_1 = \cdots = \alpha(p_k)s_k.$$

It follows that  $\text{ord}_2(\alpha(p_1)) = \cdots = \text{ord}_2(\alpha(p_k))$ . Therefore, we deduce that  $\alpha(b) = \text{lcm}(\alpha(p_1), \dots, \alpha(p_k))$ . If  $b$  is even, Proposition 2 implies that  $\alpha(2) = 1$  and so the equality  $\text{ord}_2(\alpha(p_1)) = \cdots = \text{ord}_2(\alpha(p_k))$  yields that  $\alpha(p_i)$  ( $i = 1, \dots, k$ ) are odd. Hence  $\alpha(b)$  is odd.  $\square$

## 6. The case $n = p^a q^b$

In this section we specialize the results of Theorem 1 in case where  $n = p^a q^b$ , where  $p, q$  are distinct primes and  $a, b$  positive integers.

**Proposition 4.** Let  $n = p^a q^b$ , where  $p, q$  are odd primes and  $a, b$  positive integers with  $a$  odd and  $b$  even. Let  $(x, y) \in \mathbb{Z}^2$  be a nontrivial integer solution of (1) with  $x \notin \Sigma_n \cup \Lambda_n$ . Then we have  $(x, p) = (9n, 5), (9801n, 29)$  or

$$x = \frac{p^a q^{2c} (r^2 + q^{(b-2c)2})^2}{4r(r^2 - q^{(b-2c)2})}, \quad 0 \leq 2c < b,$$

where  $r$  is a primitive element of  $K = \mathbb{Q}(\sqrt{p}, \sqrt{2})$ ,  $r, (r \pm q^{b-2c})/\sqrt{2} \in O_K$  with  $N_K(r) = N_K((r \pm q^{b-2c})/\sqrt{2}) = q^{(b-2c)4}$ ,  $q \equiv \pm 1 \pmod{8}$  and  $(p/q) = 1$ . Moreover, the index form equation  $I_K(x_2, x_3, x_4) = \pm 2^7 n^3 / \sqrt{D_K}$  has a solution.

**Proof.** We follow the notations of Theorem 1. By Theorem 1,  $\text{ord}_p(d) = \text{ord}_p(n)$ . Thus  $d = p^a q^{2c}$ , where  $0 \leq 2c \leq b$ . Suppose first that  $2c < b$ . Since  $wvu = p$ , we have the following cases:

1.  $(w, u, v) = (p, 1, 1)$ . Thus  $K = \mathbb{Q}(\sqrt{2^e p})$  which is a contradiction.
2.  $(w, u, v) = (1, p, 1)$ . If  $e = 0$ , then  $K = \mathbb{Q}(\sqrt{p})$  which is a contradiction. It follows that  $e = 1$  and  $K = \mathbb{Q}(\sqrt{p}, \sqrt{2})$ . Theorem 1 gives  $(2p/q) = (2/q) = (-1/p) = 1$ . It follows that  $q \equiv \pm 1 \pmod{8}$ ,  $p \equiv 1 \pmod{4}$ ,  $(p/q) = 1$ .
3.  $(w, u, v) = (1, 1, p)$ . If  $e = 0$ , then we have again  $K = \mathbb{Q}(\sqrt{p})$  which is a contradiction. Thus,  $e = 1$  and  $K = \mathbb{Q}(\sqrt{p}, \sqrt{2})$ . From Theorem 1, we have  $(2p/q) = (2/q) = 1$ . Hence  $q \equiv \pm 1 \pmod{8}$  and  $(p/q) = 1$ .

Suppose next  $2c = b$ . Then  $d = n$  and  $wuv = p$ . If  $w = p$ , then  $K = \mathbb{Q}(\sqrt{2^e p})$  which is a contradiction. Hence  $uv = p$ . By Lemma 3, we have  $x = nA^2$  and  $x^2 - n^2 = n^2 p D^2$ , where  $A, D \in \mathbb{Z}$ . Thus  $A^4 - 1 = p D^2$ . By Theorem 3, we have either  $p = 5$  and  $x = 9n$  or  $p = 29$  and  $x = 9801n$ .

By Theorem 1, there is  $\eta \in O_K$  satisfying  $\Delta(\eta) = 2^{14} n^6$ . Then  $I_K(\eta) = 2^7 n^3 / \sqrt{D_K}$  and so, the index form equation  $I_K(x_2, x_3, x_4) = \pm 2^7 n^3 / \sqrt{D_K}$  has a solution.  $\square$

**Proposition 5.** Let  $n = p^a q^b$ , where  $p, q$  are odd primes and  $a, b$  odd positive integers. Let  $(x, y) \in \mathbb{Z}^2$  be a nontrivial integer solution of (1) with  $x \notin \Sigma_n \cup \Lambda_n$ . Then we have one of the following cases:

1.  $x = n\alpha, n(2\alpha^2 - 1)$ , where  $(\alpha, \beta)$  is the fundamental solution of the Pell equation  $x^2 - pqy^2 = 1$ .

2. 
$$x = \frac{p^a q^{2c} (r^2 + q^{(b-2c)2})^2}{4r(r^2 - q^{(b-2c)2})}, \quad 0 \leq 2c < b,$$
 where  $r$  is a primitive integer element of  $K = \mathbb{Q}(\sqrt{p}, \sqrt{2})$  with  $N_K(r) = p^{(a-2c)4}$ ,  $q \equiv \pm 1 \pmod{8}$ ,  $(p/q) = 1$  and  $(r \pm q^{b-2c})/\sqrt{2} \in O_K$  with  $N_K((r \pm q^{b-2c})/\sqrt{2}) = q^{(b-2c)4}$ .
3.  $x = nT_l$ , where  $T_l + U_l\sqrt{q} = (T + U\sqrt{q})^l$ ,  $T + U\sqrt{q}$  is the fundamental solution of the Pell equation  $x^2 - qy^2 = 1$  and  $l \mid (p - (q/p))/2$ . Further,  $(p/q) = (-q/p) = 1$ .

Moreover, the index form equation  $I_K(x_2, x_3, x_4) = \pm 2^7 n^3 / \sqrt{D_K}$  has a solution.

**Proof.** We follow the notations of Theorem 1. Suppose first that  $d_1 = p$ . By Theorem 1,  $\text{ord}_p(d) = \text{ord}_p(n)$ . Thus  $d = p^a q^{2c}$ , where  $0 \leq 2c < b$ . On the other hand,  $wvu = p$  and so we have the following cases:

1.  $(w, u, v) = (p, 1, 1)$ . Thus  $K = \mathbb{Q}(\sqrt{2^e p})$  which is a contradiction.
2.  $(w, u, v) = (1, p, 1)$ . If  $e = 0$ , then  $K = \mathbb{Q}(\sqrt{p})$  which is a contradiction. It follows that  $e = 1$  and  $K = \mathbb{Q}(\sqrt{p}, \sqrt{2})$ . Theorem 1 gives  $(2p/q) = (2/q) = (-q/p) = 1$ . It follows that  $q \equiv \pm 1 \pmod{8}$  and  $(p/q) = (-q/p) = 1$ .
3.  $(w, u, v) = (1, 1, p)$ . If  $e = 0$ , then we have again  $K = \mathbb{Q}(\sqrt{p})$  which is a contradiction. Thus,  $e = 1$  and  $K = \mathbb{Q}(\sqrt{p}, \sqrt{2})$ . From Theorem 1, we have  $(2p/q) = (2/q) = (q/p) = 1$ . Hence  $q \equiv \pm 1 \pmod{8}$  and  $(p/q) = (q/p) = 1$ .

If  $d_1 = q$ , then we have the same results by changing the role of  $p$  and  $q$ .

Suppose now that  $d_1 = pq$ . By Theorem 1, we have  $d = n$  and  $wuv = pq$ . Thus, we have the following cases:

1.  $w = 1, uv = pq$ . By Lemma 3, we have  $x = nA^2$  and  $x^2 - n^2 = n^2 pqD^2$ , where  $A, D \in \mathbb{Z}$ . Thus  $A^4 - 1 = pqD^2$ . By Theorem 2, we have  $x = n\alpha, n(2\alpha^2 - 1)$ , where  $\alpha + \beta\sqrt{pq}$  is the fundamental solution of the equation  $x^2 - pqy^2 = 1$ .
2.  $w = pq, u = v = 1$ . By Theorem 1,  $K = \mathbb{Q}(\sqrt{2^e pq})$  which is a contradiction.
3.  $w = p, uv = q$ . Lemma 3 implies that  $x = npA^2$  and  $x^2 - n^2 = n^2 qD^2$ , where  $A, D \in \mathbb{Z}$ . Further, we have  $p^2 A^4 - qD^2 = 1$ . Let  $T + U\sqrt{q}$  be the fundamental solution of the Pell equation  $x^2 - qy^2 = 1$  and put  $T_k + U_k\sqrt{q} = (T + U\sqrt{q})^k, k \geq 1$ . By Theorem 3 and Proposition 1, we have  $pA^2 = T_l$  and  $l$  divides the quantity  $(p - (q/p))/2$ . Hence  $x = nT_l$ . Furthermore, Theorem 1 implies that  $(p/q) = (-q/p) = 1$ .
4.  $w = q, uv = p$ . Let  $V + W\sqrt{p}$  be the fundamental solution of the Pell equation  $x^2 - py^2 = 1$  and put  $V_k + W_k\sqrt{p} = (V + W\sqrt{p})^k, k \geq 1$ . We deduce, as in the previous case, that  $x = nV_h$  with  $h$  divides  $(q - (p/q))/2$  and  $(-p/q) = (q/p) = 1$ .

Finally, Theorem 1 implies that the equation  $I_K(x_2, x_3, x_4) = \pm 2^7 n^3 / \sqrt{D_K}$  has a solution.  $\square$

We remark that all the conjugates of  $r$  give the same  $x$ . So it is enough to determine one of them. The solvability of the index form equation  $I_K(x_2, x_3, x_4) = \pm 2^7 n^3 / \sqrt{D_K}$  can be checked using Proposition 1. We use MAPLE for testing the solvability of the associated Pell equations. In case where all equations have solution we try to check the solvability of the system. If we cannot easily decide, then we continue by solving the corresponding unit equations.

**Example 1.** The integer points of the elliptic curve  $E_{469\,409}$  are

$$(x, y) = (0, 0), (\pm 469\,409, 0), (9\,628\,609, \pm 29\,842\,047\,480).$$

**Proof.** The mwrank program of Cremona gives that  $E_{469\,409}$  has rank 2. We have  $469\,409 = 41 \cdot 107^2$ . So, we can use Proposition 4. Since  $107 \equiv 3 \pmod{8}$ , the integer solutions of  $E_{469\,409}$  are  $(0, 0)$ ,  $(\pm 469\,409, 0)$  and the couples obtained in steps 1, 2 and 3. We have  $\Lambda_{469\,409} = \emptyset$  and  $\Sigma_{469\,409}$  gives the solutions  $(9\,628\,609, \pm 29\,842\,047\,480)$ .  $\square$

**Example 2.** The integer points of the elliptic curve  $E_{1127}$  are

$$(x, y) = (0, 0), (\pm 1127, 0).$$

**Proof.** The mwrank program of Cremona implies that the rank of  $E_{1127}$  is 1. We have  $1127 = 23 \cdot 7^2$ . Thus, we shall use Proposition 4. We easily deduce that the elements of  $\Sigma_{1127}$  and  $\Lambda_{1127}$  do not give any no trivial solution. Next, since  $(23/7) = 1$  and  $7 \equiv -1 \pmod{8}$ , we have to check the solvability of the index form equation  $I_K(x_2, x_3, x_4) = \pm 2^7 1127^3 / \sqrt{D_K}$ , where  $K = \mathbb{Q}(\sqrt{23}, \sqrt{2})$ . By Section 4,  $D_K = 2^8 23^2$  and so we have the equation  $I_K(x_2, x_3, x_4) = \pm 2^3 23^2 7^6$ . We shall apply Proposition 1. The triples  $(F_1, F_2, F_3)$  satisfying  $-2F_1 + 23F_2 = F_3$  and  $F_1 F_2 F_3 = \pm 2^3 23^2 7^6$  are:

$$(e1127, -e98, -e4508), \quad (e2254, e98, -e2254), \quad (e1127, e196, e2254),$$

where  $e = \pm 1$ . For the first triple we have the following two systems of Pell equations:

$$x^2 - 23y^2 = e2254, \quad 2z^2 - y^2 = -e98, \quad 46z^2 - x^2 = -e4508.$$

Since the equation  $x^2 - 23y^2 = 2254$  has no solution, the system with  $e = 1$  has no solution. For  $e = -1$ , we see that every equation has a solution. So, we investigate the system. If  $(x, y, z) \in \mathbb{Z}^3$  is a solution, then the first and second equation imply that  $x$  and  $y$  are even and so we obtain  $4 \mid 2254$  which is a contradiction. Hence the system has no solution. For the second triple we obtain the following two systems:

$$x^2 - 23y^2 = e4508, \quad 2z^2 - y^2 = -e98, \quad 46z^2 - x^2 = e2254.$$

The equations  $x^2 - 23y^2 = 4508$  and  $46z^2 - x^2 = -2254$  have no solution. Thus, the two system have no solution. Finally, the last triple yields the following two systems:

$$x^2 - 23y^2 = e2254, \quad 2z^2 - y^2 = e196, \quad 46z^2 - x^2 = e2254.$$

Since  $x^2 - 23y^2 = 2254$  and  $46z^2 - x^2 = -2254$  have no solution the system has no solution. Therefore, the index form equation has no solution and so, the only integer points on  $E_{1127}$  are the couples  $(0, 0)$  and  $(\pm 1127, 0)$ .  $\square$

**Example 3.** The integer solutions to the equation  $E_{16241}$  are

$$(x, y) = (0, 0), (\pm 16241, 0).$$

**Proof.** The mwrank program of Cremona implies that the rank of  $E_{16241}$  is 2. We have  $16241 = 149 \cdot 109$  and  $109, 149$  are primes with  $149 \equiv 109 \equiv 5 \pmod{8}$  and  $(149/109) = (109/149) = -1$ . We shall use Proposition 5. We easily see that the fundamental solution  $(\gamma, \delta)$  to  $x^2 - 16241y^2 = 1$  does not give integer point  $(x, y)$  on  $E_{16241}$  with  $x = 16241\gamma$  or  $16241(2\gamma^2 - 1)$ . Finally, we easily verify that  $\Sigma_{16241}$  and  $\Lambda_{16241}$  do not give any integer point of  $E_{16241}$ .  $\square$

**Example 4.** The integer solutions to the equation  $E_{11659}$  are

$$(x, y) = (0, 0), (\pm 11659, 0).$$

**Proof.** By the mwrank program of Cremona the rank of  $E_{11659}$  is 2. We have  $11659 = 131 \cdot 89$  and  $131, 89$  are primes with  $89 \equiv 1 \pmod{8}$ ,  $131 \equiv 3 \pmod{8}$ ,  $(89/131) = (131/89) = (-131/89) = 1$  and  $(-89/131) = -1$ . We apply Proposition 5. We easily see that the sets  $\Lambda_{11659}$  and  $\Sigma_{11659}$  give no nontrivial integer point on  $E_{11659}$ . The fundamental solution  $(\gamma, \delta)$  to  $x^2 - 11659y^2 = 1$  has  $\gamma = 161558523925453228690$ . We see that there is no integer point  $(x, y)$  to  $E_{11659}$  with  $x = 11659\gamma$

or  $11659(2\gamma^2 - 1)$ . Since  $(89/131) = (-131/89) = 1$ , we consider the set  $\Pi = \{1, 2, 4, 11, 22, 44\}$  and the fundamental solution  $10610 + 927\sqrt{131}$  to  $x^2 - 131y^2 = 1$ . We determine the quantities  $T_l$  with  $T_l + U_l\sqrt{131} = (10610 + 927\sqrt{131})^l$ ,  $l \in \Pi$ , and we deduce that there is not an integer point  $(x, y)$  on  $E_{11659}$  with  $x = 11659T_l$ ,  $l \in \Pi$ .

Finally, since  $89 \equiv 1 \pmod{8}$  and  $(131/89) = 1$  we have to examine the solvability of the index form equation  $I_K(x_2, x_3, x_4) = \pm 2^3 89^3 131^2$ , where  $K = \mathbb{Q}(\sqrt{2}, \sqrt{131})$ . We shall use Proposition 1. The triples  $(F_1, F_2, F_3) = (e89 \cdot 131, e2 \cdot 89, e2^2 \cdot 89 \cdot 131)$ , where  $e = \pm 1$  satisfy  $F_1 F_2 F_3 = 2^3 89^3 131^2$  and  $2F_1 + 131F_2 = F_3$ . The associate systems of Pell equations are:

$$x^2 - 131y^2 = -e2 \cdot 89 \cdot 131, \quad 2z^2 - y^2 = e2 \cdot 89, \quad 2 \cdot 131z^2 - x^2 = e2^2 \cdot 89 \cdot 131.$$

The first and second equation imply that  $x$  and  $y$  are even and so we obtain  $2 \mid 89 \cdot 131$  which is a contradiction. Thus, the two systems have no solution. The result follows.  $\square$

Let  $n = 2^a p^b$ , where  $p$  is an odd prime and  $a, b$  are positive integers not both even. In this case, we developed in [7], an algorithm for the calculation of the integer solutions of (1). Here, we give an alternative method using the above ideas.

Let  $a \geq 3$ . If  $a$  is even, then we put

$$S_n = \{p^b(2^{a-2} + 1)^2, p^b(2^{a-2} - 1)^2, p^b(2^{2(a-1)} + 1), 2^{a-1}(p^{2b} + 1)\}.$$

Let  $a$  be odd. If  $b$  is even, then we set

$$S_n = \{2^{a-2}(p^b + 1)^2, p^b(2^{2(a-1)} + 1)\}.$$

If  $b$  is odd, then we put

$$S_n = \{2^{a-2}(p^b + 1)^2, p^b(2^{2(a-1)} + 1), p^b(2^{a-2} + 1)^2, p^b(2^{a-2} - 1)^2\} \cup L_n,$$

where  $L_n = \{2^{a-3}25\}$  when  $(p, b) = (3, 1)$ ,  $L_n = \{3^{b-1}25\}$  when  $(p, a) = (3, 3)$  and  $L_n = \emptyset$  otherwise. Finally, for  $a = 2$  we put  $S_n = \{2(p^{2b} + 1), p^b 5\}$  and for  $a = 1$ ,  $S_n = \{(p^b + 1)^2/2, (p^b - 1)^2/2\}$ .

**Proposition 6.** Let  $n = 2^a p^b$ , where  $p$  is an odd prime and  $a, b$  are positive integers not both even. Let  $(x, y) \in \mathbb{Z}^2$  be an integer solution of (1) with  $x > n$  and  $x \notin S_n \cup L_n$ . Then, either  $a$  is even,  $b$  is odd and  $(p, x) = (5, 9n), (29, 9801n)$  or  $a, b$  are odd and we have one of the following cases:

1.  $(x, p) = (2n, 3), (49n, 3), (8n, 7)$ .
2.  $x = nA_l$ , where  $A_l + B_l\sqrt{2} = (3 + 2\sqrt{2})^l$  and  $l$  is a positive integer with  $l \mid (p - (2/p))/2$ .

**Proof.** By [7, Lemma 1], we have  $x = nz$  and  $z$  is a positive integer. Then  $(y/n)^2 = zn(z^2 - 1)$ . We have the following cases:

1.  $a$  odd,  $b$  even. Then  $y_1^2 = 2z(z^2 - 1)$  with  $y_1 = y/n2^{(a-1)/2}p^{b/2}$ . If  $z$  is even, then  $z^2 - 1 = L^2$ , where  $L \in \mathbb{Z}$ , and so  $z = 1$  which is a contradiction. Suppose that  $z$  is odd. Then  $z = M^2$  and  $z^2 - 1 = 2N^2$ , where  $M, N \in \mathbb{Z}$ . It follows that  $M^4 - 2N^2 = 1$ . By [4], this equation has no nontrivial solution.

2.  $a$  even,  $b$  odd. We have  $y_1^2 = pz(z^2 - 1)$  with  $y_1 \in \mathbb{Z}$ . If  $p \mid z$ , then we obtain, as in the previous case, a contradiction. Suppose next that  $p \nmid z$ . Then  $z = M^2$  and  $z^2 - 1 = pN^2$ , where  $M, N \in \mathbb{Z}$ . Thus  $M^4 - pN^2 = 1$ . By Theorem 3,  $p = 5, 29$  and  $x = 9n, 9801n$ , respectively.

3.  $a$  odd,  $b$  odd. We have  $y_1^2 = 2pz(z^2 - 1)$  with  $y_1 \in \mathbb{Z}$ . If  $2p \mid z$ , then we obtain, as above, a contradiction. If  $2 \mid z$  and  $p \nmid z$ , then we have  $z = 2M^2$  and  $z^2 - 1 = pN^2$ , where  $M, N \in \mathbb{Z}$ . It follows that  $4M^4 - pN^2 = 1$ . By Theorem 4, we have  $(p, z) = (3, 2), (7, 8)$ . If  $p \mid z$  and  $2 \nmid z$ , then  $z = pM^2$  and  $z^2 - 1 = 2N^2$ , where  $M, N \in \mathbb{Z}$ , and so  $p^2M^4 - 2N^2 = 1$ . By Theorem 5 and Proposition 2, we have  $z = A_l$ , where  $A_l + B_l\sqrt{2} = (3 + 2\sqrt{2})^l$  and  $l$  is a positive integer with  $l \mid (p - (2/p))/2$ . Finally, if  $p \nmid z$

and  $2 \nmid z$ , then  $z = M^2$  and  $z^2 - 1 = 2pN^2$ , where  $M, N \in \mathbb{Z}$ . Thus  $M^4 - 2pN^2 = 1$  and so Theorem 3 implies that  $p = 3$  and  $z = 49$ .  $\square$

The integer points  $(x, y)$  of  $E_n$ , in the above case, with  $x < 0$  are given by [7, Lemma 2].

**Example 5.** The integer points of  $E_6$  are

$$(x, y) = (0, 0), (\pm 6, 0), (-3, \pm 9), (-2, \pm 8), (12, \pm 36), (18, \pm 72), (294, \pm 5040).$$

**Proof.** This equation has been solved already in [7]. We shall solve it using Proposition 6. First, we remark that  $(294, \pm 5040)$  and  $(12, \pm 36)$  are integers points on  $E_6$ . Next, since  $(3 - (2/3))/2 = 2$ , we deduce that  $A_1 = 3$  and  $A_2 = 17$  and so we obtain the integer point  $(18, 72)$ . Further, we have  $S_6 = \{2, 8\}$ , whence we do not obtain any integer point. Finally, we easily verified that the integer points  $(x, y)$  with  $x < 0$  are:  $(-6, 0), (-3, \pm 9), (-2, \pm 8)$ .  $\square$

## 7. The general case

In this section we present an algorithm for the solution of (1) in case where  $n$  has at least two prime divisors. Using Theorem 1, we obtain the following algorithm:

### Algorithm.

**Input:** An integer  $n > 1$  which is not a square or of the form  $p^a$ , where  $p$  is a prime and  $a$  an integer  $> 0$ .

**Output:** The integer solutions of (1).

1. Determine the integer solutions  $(x, y)$  of (1) with  $x \in \Sigma_n$ .
2. Determine the integer solutions  $(x, y)$  of (1) with  $x \in \Lambda_n$ .
3. Determine the set  $D$  of positive divisors  $d$  of  $n$  having primes in its prime decomposition with odd exponent and for every such prime  $p$  we have  $\text{ord}_p(d) = \text{ord}_p(n)$ .
4. For every  $d \in D$  we put  $m = n/d$  and  $d = d_1 \delta^2$ , where  $d_1, \delta \in \mathbb{Z}$  and  $d_1$  is square-free. We determine the set  $D(d)$  of quadruples of integers  $(w, u, v, e)$  with  $wuv = d_1$ ,  $w, u, v > 0$  and  $e \in \{0, 1\}$  such that we have

$$\mathbb{Q}(\sqrt{uv}, \sqrt{2^e wv}) = \mathbb{Q}(\sqrt{uv}, \sqrt{2^e wu}) = \mathbb{Q}(\sqrt{2^e wu}, \sqrt{2^e wv}),$$

$[\mathbb{Q}(\sqrt{uv}, \sqrt{2^e wv}) : \mathbb{Q}] = 4$  and the following equalities are satisfied:

- For every odd prime divisor  $p$  of  $m$  we have

$$(wu2^e/p) = 1, \quad (wv2^e/p) = 1.$$

- For every odd prime divisor  $p$  of  $w$  we have

$$(mu2^e/p) = 1, \quad (-mv2^e/p) = 1.$$

- For every odd prime divisor  $p$  of  $u$  we have

$$(-wm/p) = 1, \quad (-mv2^{e+1}/p) = 1.$$

- For every odd prime divisor  $p$  of  $v$  we have

$$(wm/p) = 1, \quad (mu2^{e+1}/p) = 1.$$

5. For every  $d \in D$ , determine the subset  $G(d)$  of  $D(d)$  which contains the quadruples  $(w, u, v, e) \in D(d)$  such that the equation  $I_K(x_2, x_3, x_4) = \pm 2^7 n^3 / \sqrt{D_K}$ , where  $K = \mathbb{Q}(\sqrt{uv}, \sqrt{2^e wv})$ , has an integer solution.
6. For every  $d \in D$  and for every  $\lambda = (w, u, v, e) \in G(d)$  determine maximal sets  $R_{d,\lambda}$  and  $S_{d,\lambda}$  of pairwise nonassociate algebraic integers of  $\mathbb{Q}(\sqrt{uv}, \sqrt{2^e wv})$  with norm  $m^4$  and  $4m^4$ , respectively.
7. For every  $d \in D$  and for every  $\lambda = (w, u, v, e) \in G(d)$  determine the units  $\eta$  in  $\mathbb{Q}(\sqrt{uv}, \sqrt{2^e wv})$  for which there are  $\alpha \in R_{d,\lambda}$  and  $\beta \in S_{d,\lambda}$  such that  $(\alpha\eta - m)/\beta$  is also a unit. Denote by  $W_{d,\lambda}$  the set of elements  $\alpha\eta$ .
8. For every  $d \in D$  and for every  $\lambda \in G(d)$  determine the integer solutions  $(x, y)$  of (1) with  $x > n$  and

$$x = d \frac{(m^2 + r^2)^2}{4r(r^2 - m^2)},$$

where  $r \in W_{d,\lambda}$ .

9. The integer solutions of (1) are the couples  $(0, 0)$ ,  $(\pm n, 0)$  and the couples obtained in steps 1, 2 and 8.

**Remark 1.** If  $d = n$  and  $G(n) \neq \emptyset$ , then the solutions given by steps 6, 7 and 8 can be also obtained by  $x = nWA^2$ , where  $(A, B)$  are the integer solutions of  $w^2X^4 - uvY^2 = 1$ , for every  $(w, u, v, e) \in G(n)$ .

**Remark 2.** In step 6, if  $\sqrt{2} \in \mathbb{Q}(\sqrt{uv}, \sqrt{2^e wv})$ , then we can alternatively take  $S_{d,\lambda}$  to be a maximal set of pairwise nonconjugate elements of  $R_{d,\lambda}$  and in step 7 to consider the units  $\eta$  in  $\mathbb{Q}(\sqrt{uv}, \sqrt{2^e wv})$  for which there are  $\alpha \in R_{d,\lambda}$  and  $\beta \in S_{d,\lambda}$  such that  $(\alpha\eta - m)/\beta\sqrt{2}$  is also a unit.

**Example 6.** The integer points of  $E_{30}$  are

$$(x, y) = (0, 0), (\pm 30, 0), (-6, \pm 72), (-20, \pm 100), (45, \pm 225), (150, \pm 1800).$$

**Proof.** The mwrank program of Cremona implies that the rank of  $E_{30}$  is 1. We easily obtain that the set  $\Sigma_{30}$  does not give any nontrivial integer point and the set  $\Lambda_{30}$  gives the points  $(-6, \pm 72)$  and  $(-20, \pm 100)$ . For  $d = 30$  we have the quadruples:  $(w, u, v, e) = (1, 1, 30, 1), (1, 2, 15, 0), (5, 3, 2, 1), (5, 6, 1, 0), (6, 5, 1, 1)$ .

We shall apply Remark 1. The two first quadruples correspond to the equation  $x^4 - 30y^2 = 1$ . Using Theorem 2 we deduce that this equation has no nontrivial solution. The other two give the equation  $25x^4 - 6y^2 = 1$ . The fundamental solution to  $x^2 - 6y^2 = 1$  is  $(5, 2)$ . So, Corollary 1 yields that  $(1, 2)$  is the only nontrivial solution of  $25x^4 - 6y^2 = 1$ . This gives the integer point  $(150, \pm 1800)$  of  $E_{30}$ . Similarly, for the last quadruple we examine the equation  $36x^4 - 5y^2 = 1$ . By Corollary 1, this equation has no nontrivial solution.

Let  $d = 15$ . The only quadruple satisfying the equalities of step 7 is  $(w, u, v, e) = (3, 5, 1, 0)$  which yields the field  $K = \mathbb{Q}(\sqrt{5}, \sqrt{3})$ . In order to apply Proposition 1, we write  $K = \mathbb{Q}(\sqrt{15}, \sqrt{3})$  and so we have Case 5 with  $l = 3$ ,  $a_1 = 5$  and  $b_1 = 1$ . The triples  $(F_1, F_2, F_3) = (-e24, e120, e20)$ , where  $e = \pm 1$ , satisfy the equalities  $5F_1 + F_2 = 12F_3$  and  $F_1F_2F_3 = \pm 2^8 3^2 5^2 = \pm 2^7 30^3 / \sqrt{D_K}$ . We have the following systems:

$$3x^2 - y^2 = -e24, \quad 3z^2 - 5y^2 = e120, \quad z^2 - 5x^2 = e80.$$

Using MAPLE, we see that all the above Pell equations have a solution. Thus, we cannot deduce that the associated index form equation has no solution and so, we continue to the next steps of our algorithm. We use the computational system KANT 2.5 and we obtain the following integral basis for the field  $K$ :

$$\omega_0 = 1, \quad \omega_1 = \theta, \quad \omega_2 = (-2 - 2\theta + \theta^2)/4, \quad \omega_3 = (-4 - 2\theta + \theta^3)/8,$$



Table 1

$([-16, 22, 3, -12], [1, 2, 2, 1]),$	$([3, 13, -2, -8], [-1, -2, -4, 2]),$
$([-11, 15, 2, -8], [-1, -1, -4, 2]),$	$([-4, 7, 1, -4], [2, 3, 6, -1]),$
$([-2, -1, -3, -2], [33, 29, 47, 25]),$	$([0, 2, -1, -2], [10, 9, 15, 7]),$
$([2, 3, -1, -2], [1, 0, -1, 3]),$	$([1, 3, 0, -2], [2, 3, 7, -2]),$
$([-4, 4, 1, -2], [0, 0, -1, 1]),$	$([-1, -3, -2, 0], [23, 20, 32, 18]),$
$([-1, 1, -2, 0], [-5, -12, -32, 18]),$	$([0, -1, -1, 0], [8, 6, 8, 9]),$
$([0, 0, -1, 0], [1, -2, -8, 9]),$	$([0, 0, 1, 0], [1, 3, 8, -5]),$
$([0, 1, 1, 0], [-6, -5, -8, -5]),$	$([1, -1, 2, 0], [7, 13, 32, -14]),$
$([1, 3, 2, 0], [-21, -19, -32, -14]),$	$([0, -2, -3, 2], [-8, -18, -47, 25]),$
$([4, -4, -1, 2], [2, 1, 1, 3]),$	$([2, -3, -1, 2], [-3, -6, -15, 7]),$
$([3, -3, 0, 2], [-4, -4, -7, -2]),$	$([-2, -3, 1, 2], [1, 1, 1, 1]),$
$([0, -6, 1, 4], [-3, -3, -6, -1]),$	$([11, -15, -2, 8], [3, 2, 4, 2]),$
$([-3, -13, 2, 8], [3, 3, 4, 2]),$	$([-4, -19, 3, 12], [0, 0, -2, 1])$

where  $\theta = \sqrt{3} + \sqrt{5}$ . We represent an algebraic integer of  $K$ ,  $z = \sum_{i=0}^3 z_i \omega_i$ , where  $z_i \in \mathbb{Z}$  ( $i = 0, 1, 2, 3$ ), by  $z = [z_0, z_1, z_2, z_3]$ . A maximal set of pairwise nonassociate algebraic integers of  $K$  with norm 16 and 64 is given by the elements  $\alpha = [2, 0, 0, 0]$  and  $\beta = [32, -110, 0, 64]$ , respectively. Using KANT 2.5 we obtain the solutions of the unit equation

$$\alpha U - \beta V = 2$$

which are listed in Table 1.

The above solutions yield the integer point  $(45, \pm 225)$  of  $E_{30}$ . For if  $(u, v) = ([0, 0, 1, 0], [1, 3, 8, -5])$ , then putting  $r = 2u$  we obtain

$$x = 15 \frac{(4 + r^2)^2}{4r(r^2 - 4)} = 45.$$

For  $d = 1, 2, 3, 5, 6, 10$  the set  $G(d)$  is empty.  $\square$

**Example 7.** The integer points of  $E_{110}$  are

$$(x, y) = (0, 0), (\pm 110, 0), (-90, \pm 600), (-11, \pm 363), (1100, \pm 36300).$$

**Proof.** The mwrank program implies that the rank of  $E_{110}$  is 1. It is easily seen that  $\Sigma_{110}$  does not give any nontrivial integer point and  $\Lambda_{110}$  yields the points  $(-11, \pm 363)$  and  $(-90, \pm 600)$ . For  $d = 110$  we have:  $(w, u, v, e) = (1, 1, 110, 1), (1, 2, 55, 0), (2, 5, 11, 1), (11, 10, 1, 1), (11, 5, 2, 0)$  and  $(10, 11, 1, 0)$ .

The first two quadruples give the equation  $x^4 - 110y^2 = 1$ . Using Theorem 2, we see that the fundamental solution  $(21, 2)$  to  $x^2 - 110y^2 = 1$  does not yield a solution for  $x^4 - 110y^2 = 1$  and consequently an integer point for  $E_{110}$ . The third quadruple corresponds to the equation  $4x^4 - 55y^2 = 1$ . The fundamental solution of  $x^2 - 55y^2 = 1$  is  $(89, 12)$  and so, Corollary 1 implies that  $x^4 - 110y^2 = 1$  has no nontrivial solution. The other two quadruples yield the equation  $11^2x^4 - 10y^2 = 1$ . The fundamental solution to  $x^2 - 10y^2 = 1$  is  $(19, 6)$ . By Corollary 1, we see that  $11^2x^4 - 10y^2 = 1$  does not have a nontrivial solution. The last quadruple gives the equation  $10^2x^4 - 11y^2 = 1$ . The fundamental solution to  $x^2 - 11y^2 = 1$  is  $(10, 3)$ . By Corollary 1 the equation  $10^2x^4 - 11y^2 = 1$  has only the solution  $(x, y) = (1, 3)$ . Thus, we obtain the integer point  $(1100, \pm 36300)$  of  $E_{110}$ .

For  $d < 110$ , we get  $D(55) = \{(5, 11, 1, 1)\}$  and  $D(d) = \emptyset$  for  $d \neq 55, 110$ . Thus, we obtain the field  $K = \mathbb{Q}(\sqrt{10}, \sqrt{11})$ . The triples  $(F_1, F_2, F_3)$  satisfying the equalities  $-10F_1 + 11F_2 = F_3$  and  $F_1F_2F_3 = \pm 2^6 5^2 11^2$  are  $(-e22, e20, e440)$  and  $(e44, e20, -e220)$ , where  $e = \pm 1$ . For the first triple we have the following systems:

$$x^2 - 11y^2 = -e44, \quad 2z^2 - 5y^2 = e20, \quad 22z^2 - 5x^2 = e440.$$

Since the equation  $22z^2 - 5x^2 = \pm 440$  has no solution, the above systems have no solution. The second triple yield the systems

$$x^2 - 11y^2 = e88, \quad 2z^2 - 5y^2 = e20, \quad 22z^2 - 5x^2 = -e220.$$

The equations  $x^2 - 11y^2 = 88$  and  $22z^2 - 5x^2 = 220$  do not have solution and hence the two systems do not have solution. Thus, the associated index form equation has also no solution. The result follows.  $\square$

**Example 8.** The integer points of  $E_{255}$  are

$$(x, y) = (0, 0), (\pm 255, 0), (-225, \pm 1800), (289, \pm 2312), (4080, \pm 260100).$$

**Proof.** By the mwrank program, the rank of  $E_{255}$  is 1. We have  $n = 255 = 5 \cdot 3 \cdot 17$ . We easily see that  $\Sigma_{255}$  gives the integer point  $(289, \pm 2312)$  and  $\Lambda_{255}$  gives  $(-225, \pm 1800)$ .

For  $d = 255$  we have the quadruples:  $(w, u, v, e) = (1, 1, 255, 1), (1, 17, 15, 0)$ . The two quadruples give the equation  $x^4 - 255y^2 = 1$ . The fundamental solution of  $x^2 - 255y^2 = 1$  is  $(16, 1)$ . By Theorem 2, we obtain that the only nontrivial solution of  $x^4 - 255y^2 = 1$  is  $(4, 1)$  which gives the integer points  $(x, y) = (4080, \pm 260100)$  on  $E_{255}$ . Furthermore, we deduce that  $E(d) = \emptyset$  for  $d < 255$ .  $\square$

**Example 9.** The integer points of  $E_{8428033}$  are

$$(x, y) = (0, 0), (\pm 8428033, 0).$$

**Proof.** The mwrank program implies that the rank of  $E_{8428033}$  is  $\leq 2$ . We have  $n = 337 \cdot 281 \cdot 89 = 8428033$ . We easily deduce that the sets  $\Sigma_n$  and  $\Lambda_n$  do not provide us with nontrivial integer points of  $E_{8428033}$ . For  $d = 8428033$  we have the quadruple  $(w, u, v, e) = (1, 1, 8428033, 1)$  which gives the equation  $x^4 - 8428033y^2 = 1$ . Following Theorem 2, we compute the fundamental solution of  $x^2 - 8428033y^2 = 1$  and we see that it does not give a nontrivial solution of  $x^4 - 8428033y^2 = 1$ . Thus, this equation has no nontrivial solution. Finally, we obtain  $G(d) = \emptyset$  for  $d < n$ . The result follows.  $\square$

**Example 10.** The integer points of  $E_{1131}$  are

$$(x, y) = (0, 0), (\pm 1131, 0), (-117, \pm 12168), (10933, \pm 1137032).$$

**Proof.** First note that the rank of  $E_{1131}$  is 2. We have  $n = 3 \cdot 13 \cdot 29$ . The set  $\Sigma_{1131}$  does not give nontrivial integer point of  $E_{1131}$  and the set  $\Lambda_{1131}$  yields the point  $(-117, \pm 12168)$ . Only for  $d = 1131, 377, 87$  we obtain quadruples  $(w, u, v, e)$ .

If  $d = 1131$ , then we get  $(w, u, v, e) = (1, 1, 1131, 1), (1, 13, 87, 1), (13, 29, 3, 0)$ . We shall apply Remark 1. For the first two quadruples, we have to find the solutions of  $x^4 - 1131y^2 = 1$ . The fundamental solution of the equation  $x^2 - 1131y^2 = 1$  is  $(a, b) = (268185, 10948)$  and  $\sqrt{a}, \sqrt{2a^2 - 1}$  are not integers. So Theorem 2 implies that the equation  $x^4 - 1131y^2 = 1$  has no integer solution. Thus, the first two quadruples do not give an integer point of  $E_{1131}$ . For  $(w, u, v, e) = (13, 29, 3, 0)$  we have to solve  $13^2x^4 - 87y^2 = 1$ . Using Theorem 5 and Proposition 2 we deduce that this equation has no solution and so the last quadruple does not give an integer point of  $E_{1131}$ .

For  $d = 377$  we have the quadruples  $(w, u, v, e) = (29, 13, 1, 1), (29, 1, 13, 1)$ . Both quadruples yield the field  $K = \mathbb{Q}(\sqrt{58}, \sqrt{13})$ . The triples  $(F_1, F_2, F_3) = (e348, e39, e4524)$ , where  $e = \pm 1$ , satisfy  $-13F_1 + 4 \cdot 58F_2 = F_3$  and  $F_1F_2F_3 = \pm 2^4 3^3 13^2 29^2 = \pm 2^7 1131^3 / \sqrt{D_K}$ . Thus, we have the systems

$$x^2 - 58y^2 = e348, \quad z^2 - 13y^2 = e156, \quad 58z^2 - 13x^2 = e4524.$$

For  $e = -1$ , the system has the solution  $(58, 8, 26)$ . So we continue to the next steps of our algorithm.

Table 2

[46, −30, −17, 70]
[4, 2, −1, −4]
[4, 1, 0, −1]
[28, −16, −1, 14]
[646, 236, −17, −196]
[−19, −2, 4, 0]
[−167 254, 78 936, 55 361, −193 688]
[8, 3, 0, −2]
[−116, −47, 17, 70]
[12, 5, −2, −8]
[3, 0, 0, 0]
[54, −27, −2, 24]
[−123 378, 33 167, 35 564, −92 558]
[−6, 3, 0, −2]
[−1 330 901 358, 626 185 105, 49 998 419, −561 048 396]
[129, 2, −4, 0]
[3, −1, 0, 1]
[−2 352 490 266, 1 106 839 766, 88 376 795, −991 704 638]
[−450, 219, 17, −196]

Table 3

[−78, 39, 3, −35]
[18, 7, −1, −7]
[75 450 997 426 395, 26 966 216 894 344, −1 993 938 001 999, −22 374 625 065 691]
[9 874 766 665, 3 529 245 596, −260 959 765, −2 928 313 929]
[−9, 6, 3, −13]
[−64 259, 39 638, 23 153, −92 919]
[498, 191, −13, −159]
[30, −3, −1, 3]
[−18, 9, 1, −9]
[1 856 475 641 909, 663 505 142 760, −49 060 946 289, −550 528 791 509]
[27, 10, −1, −9]
[−391, 234, 125, −515]
[4671, 1670, −125, −1389]
[−11, 6, 1, −7]
[875 811, 313 018, −23 153, −259 737]
[113, 42, −3, −35]

Set  $\theta = \sqrt{13} + \sqrt{58}$ . The computational system MAGMA gives the following integral basis for the field  $K = \mathbb{Q}(\sqrt{58}, \sqrt{13})$ :

$$\omega_0 = 1, \quad \omega_1 = \theta, \quad \omega_2 = (\theta^2 + 2\theta + 3)/4, \quad \omega_3 = (\theta^3 + 83\theta + 90)/180.$$

If  $z = \sum_{i=0}^3 z_i \omega_i$ , where  $z_0, z_1, z_2, z_3 \in \mathbb{Z}$ , is an element of  $O_K$ , then we write  $z = [z_0, z_1, z_2, z_3]$ . A maximal set of pairwise nonassociate elements of  $O_K$  with norm  $3^4$  is given by MAGMA. Its elements are listed in Table 2.

Furthermore, MAGMA provides a maximal set of pairwise nonassociate elements of  $O_K$  with norm  $4 \cdot 3^4$  given in Table 3.

Next we solve, using MAGMA, the units equations  $aU - bV = 3$ , where  $a$  is an element of Table 2 and  $b$  an element of Table 3. The equation with  $a = [12, 5, -2, -8]$  and

$$b = [75\,450\,997\,426\,395, 26\,966\,216\,894\,344, -1\,993\,938\,001\,999, -22\,374\,625\,065\,691]$$

has only the solution

$$U = [-1, 0, 0, 0],$$
$$V = [290\,305\,319\,644, -179\,246\,403\,758, -104\,979\,192\,791, 420\,989\,904\,480]$$

which yields the integer points  $(10933, \pm 1\,137\,032)$  of  $E_{1131}$ . None of the other equations gives another integer point of  $E_{1131}$ .

For  $d = 87$  we have  $(w, u, v, e) = (1, 29, 3, 0)$  which gives the field  $\mathbb{Q}(\sqrt{3}, \sqrt{87})$ . The triples  $(F_1, F_2, F_3)$  satisfying  $F_1 F_2 F_3 = \pm 2^5 3^2 13^3 29^2 = \pm 2^7 1131^3 / \sqrt{D_K}$  and  $-F_1 - 29F_2 = 12F_3$  are the triples  $(-e9048, e156, e377)$ ,  $(e4524, e156, -e754)$  and  $(e4524, -e312, e377)$ , where  $e = \pm 1$ . For the first triple we have the systems:

$$3x^2 - 29y^2 = -e9048, \quad 3z^2 - y^2 = e156, \quad 29z^2 - x^2 = e1508.$$

The two systems have no solution since the equations  $3x^2 - 29y^2 = -9048$  and  $3z^2 - y^2 = -156$  have no solution. The second triple yields the systems:

$$3x^2 - 29y^2 = e4524, \quad 3z^2 - y^2 = e156, \quad 29z^2 - x^2 = -e3016.$$

The equations  $3x^2 - 29y^2 = \pm 4524$  have no solution and so, the systems have no solution. The last triple gives the systems

$$3x^2 - 29y^2 = e4524, \quad 3z^2 - y^2 = -e312, \quad 29z^2 - x^2 = e1508$$

which have no solution, since  $3x^2 - 29y^2 = \pm 4524$  do not. The result follows.  $\square$

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